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# One-channel Roy equations revisited*

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**Abstract.** The Roy equation in the single-channel case is a nonlinear, singular integral equation for the phase shift in the low-energy region. We first investigate the infinitesimal neighborhood of a given solution, and then present explicit expressions for amplitudes that satisfy the nonlinear equation exactly. These amplitudes contain free parameters that render the non-uniqueness of the solution manifest. They display, however, an unphysical singularity at the upper end of the interval considered. This singularity disappears and uniqueness is achieved if one uses analyticity properties of the amplitudes that are not encoded in the Roy equation.

## 1 Introduction

The elastic $\pi\pi$ amplitude has recently been evaluated in the framework of chiral perturbation theory [1] to two loops [2]. The representation is valid in the low-energy region, where the center-of-mass energy of the pions is less than about 400 MeV. On the other hand, precise experimental data are presently only available above $\sim 600$ MeV. In order to connect the two regions, one may rely on a set of dispersion relations for the partial wave amplitudes; these relations are due to Roy [3]. They allow one to extrapolate the data down to threshold and to merge with the chiral expansion [4]. In the present article, we investigate this extrapolation procedure in the one-channel case.

Roy's representation [3] for the partial wave amplitudes $t_l^I$ of $\pi\pi$ scattering reads as follows:

$$\mathrm{Re}\, t_l^I(s) = k_l^I(s) + \sum_{I'=0}^{2}\sum_{l'=0}^{\infty} \fint_4^\infty \mathrm{d}x\, K_{ll'}^{II'}(s,x)\, \mathrm{Im}\, t_{l'}^{I'}(x)\ ; \quad 4 \le s \le 60, \tag{1.1}$$

where $I$ and $l$ denote isospin and angular momentum, respectively[1]. The linear subtraction polynomials $k_l^I$ are expressed in terms of the two S-wave scattering lengths. The kernels $K_{ll'}^{II'}$ contain a diagonal, singular Cauchy kernel that generates the right-hand cut in the partial wave amplitudes, as well as a logarithmically singular piece that generates the left-hand cut.

The relations (1.1) are consequences of the exact analyticity properties of the $\pi\pi$ scattering amplitude, of the Froissart bound and of crossing symmetry. We may conclude from them that the scattering amplitudes are fully determined by the imaginary parts of the partial waves, except for the two scattering lengths that play the role of subtraction constants. Combined with unitarity, (1.1) amounts to an infinite system of coupled, singular integral equations – the Roy equations – for the phase shifts in a low-energy interval from threshold $s = 4$ to a matching point $s_0 < 60$. In this framework, the phase shifts above the matching point $s_0$, the absorption parameters and the two S-wave scattering lengths are assumed to be externally assigned. The mathematical problem consists in solving the Roy equations with this input.

Soon after the original article of Roy [3] had appeared, extensive phenomenological applications were performed [5], resulting in a detailed analysis and exploitation of the experimental data on $\pi\pi$ scattering then available. For a recent review of these results we refer the reader to the article by Morgan and Pennington [6]. Parallel to these phenomenological applications, the very structure of the Roy equations was investigated. In [8], extensions of (1.1) were presented, valid in the larger range $-28 \le s \le 125.31$. Further, the manifold of solutions of Roy's equations was investigated as well, both in the single-channel [9–11] as in the coupled-channel case [12]. In the late seventies, Pool [13] provided a proof that the original, infinite set of integral equations does have at least one solution for $s_0 < 23.31$, provided that the driving terms are not too large, see also [14]. Heemskerk and Pool also numerically examined the solutions of the Roy equations, both by solving the $N$ equation [14] and by using an iterative method [15].

It emerged from these investigations that – for a given input of absorptive parts, absorption parameters and S-

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[1] We express all energies in units of $M_\pi$. Further, $\fint$ denotes a principal value integral

wave scattering lengths – there are in general many possible solutions to the Roy equations. This non-uniqueness is due to the singular Cauchy kernel in the right-hand side of (1.1). In order to investigate the uniqueness properties of the Roy system, one may – in a first step – keep only this part of the kernels, as a result of which the integral equations decouple: one is left with a single-channel problem, i.e. a single partial wave, that has, moreover, no left-hand cut. This mathematical problem was examined by Pomponiu and Wanders [9]. Investigating the infinitesimal neighborhood of a given solution, they found that the multiplicity of the solution increases by one whenever the value of the phase shift at the matching point goes through a multiple of $\pi/2$. By contrast, the number of parameters in the usual partial-wave equation increases in general by two whenever the phase shift at infinity passes through a positive integer times $\pi$, see e.g. [16,17] and references cited therein.

Atkinson and Warnock [10] investigated a one-channel problem by using $N/D$ methods that do not require linearization. Their work may be summarized as follows. First, they find that solutions of the Roy equations with $\delta(s_0) \geq -\pi/2$ are members of solution manifolds depending on $m$ parameters, where $m$ denotes the integer part of $2\delta(s_0)/\pi$. Second, these solutions may be computed through an integral equation in which the $m$ parameters appear explicitly. An exception to these statements – whose proof is rather complex – could occur if a certain Fredholm operator had unit eigenvalue. Although the relevant integral equations in [10] exhibit the $m$ parameters, it had not been shown that these are really effective: an arbitrarily chosen set may lead to ghosts.

After 1980, interest in the Roy equations waned. In recent years, it has however become clear that there are good reasons to revive these techniques. First, new $K_{l4}$ experiments are planned [7] or are already underway [18, 19]. These will provide new information on the low-energy $\pi\pi$ scattering amplitude. A reliable analysis of available low-energy data and high-energy phase shifts should then again be based on Roy equations. Second, one can establish at this point contact with CHPT, that will allow one [1,21] to gain insight into the structure of the underlying theory, the Standard Model. First steps in this program have already been performed [22,23]. Of course, this enterprise requires that one understands the structure of the Roy equations in all details. It is the aim of this and a following article [24] to fill existing gaps in this respect and to provide insights into the problem from an actual point of view, see also [25].

We concentrate in this work on the uniqueness properties of the solutions in the one-channel case, and on their singularity structure. We start by analyzing the Roy equation with the linearization method proposed in [9] and show how the emerging integral equation can be solved by transforming it into a homogeneous Hilbert problem. This method is very efficient in determining the multiplicity of solutions. It does, however, assume the existence of a solution. Furthermore, it is not easy to show that each solution of the linearized equation approximates a solution of the nonlinear one. For these reasons, we also investigate the Roy equation for a special class of inputs that allow us to find explicit solutions. These solutions do contain parameters that exhibit their non-uniqueness. We then investigate the role of the unphysical singularity that shows up in the solutions. This singularity manifests itself as a cusp in the real and imaginary parts of the amplitude at the matching point. By making use of analyticity properties of the amplitude that are not explicit in the Roy equation, we find that there is a unique solution, devoid of cusps.

Our article is organized as follows. In Sect. 2, we formulate the mathematical problem we are concerned with. In Sect. 3, we analyze the infinitesimal neighborhood of a solution. The construction of a second exact solution – once a first solution is known – is reduced to a linear problem in Sect. 4. This allows us to give in Sect. 5 a class of explicit exact solutions. In Sect. 6 we show how uniqueness is obtained by the use of analyticity properties of the amplitudes that are not encoded in the Roy equation. A summary and concluding remarks are given in Sect. 7. Details on precise mathematical formulations are relegated to Appendix A, where we also construct the general solution of the linearized problem of Sect. 2. Appendix B provides the connection with the $N/D$ method [10], and Appendix C contains the proof of the uniqueness property for an analytic input.

In order to help orient the reader, we note that the key statements in this article are summarized in propositions 1, 2, 3 and 4 in Sects. 3, 4 and 6. Propositions 1 and 3 were established long ago in [9] and [26], respectively, whereas propositions 2 and 4 are – to the best of our knowledge – new results.

## 2 The one-channel Roy equation

In order to study the non-uniqueness properties of the Roy equation, we keep only the diagonal, singular Cauchy kernel in (1.1). The partial-wave relations then decouple, and the left-hand cut in the amplitudes disappears. We therefore first explore the set of complex amplitudes $f : [4, \infty) \to \mathbb{C}$ with the following properties:
(i) In an interval $[4, s_0]$ containing the threshold $s = 4$ and a matching point $s_0$, the real part is given by a dispersion relation

$$\operatorname{Re} f(s) = a + (s-4)\frac{1}{\pi} \,⨍_4^\infty \frac{\mathrm{d}x}{x-4} \frac{\operatorname{Im} f(x)}{x-s}. \tag{2.1}$$

(ii) The imaginary part $\operatorname{Im} f$ is a given input function $A$ above $s_0$,

$$\operatorname{Im} f(s) = A(s), \qquad s \geq s_0. \tag{2.2}$$

(iii) Elastic unitarity holds below $s_0$,

$$\operatorname{Im} f(s) = \sigma(s)|f(s)|^2,\ s \in [4, s_0]; \quad \sigma(s) = [1 - 4/s]^{1/2}. \tag{2.3}$$

We relegate a precise formulation of the regularity properties of $f$ to Appendix A and simply note that, as a minimal requirement, the imaginary part Im $f$ must be continuous in $[4, s_0]$; in particular,

$$\lim_{s \nearrow s_0} \text{Im } f(s) = A(s_0). \tag{2.4}$$

Equations (2.1)–(2.4) constitute the mathematical problem that we discuss in the following: determine the amplitudes $f$ which verify these equations for given scattering length $a$ and given absorptive part $A$. This one-channel problem allows detailed analytical and numerical calculations and provides useful insight into the solutions of the coupled system considered e.g. in [4,5,12].

Elastic unitarity allows one to reduce the problem to the determination of a single real function on the real interval $[4, s_0]$, because $f$ is parametrized by its real phase shift $\delta$ below $s_0$:

$$f(s) = \frac{1}{\sigma(s)} e^{i\delta(s)} \sin\delta(s), \qquad s \in [4, s_0]. \tag{2.5}$$

We choose the normalization of $\delta$ such that it vanishes at threshold, $\delta(4) = 0$. The boundary condition (2.4) becomes

$$\sin^2\delta(s_0) = \sigma(s_0)A(s_0). \tag{2.6}$$

The parametrization (2.5) allows one to write (2.1)–(2.3) as a nonlinear, singular integral equation for the phase shift $\delta$ in the interval $s \in [4, s_0]$,

$$\frac{1}{2\sigma(s)} \sin(2\delta(s)) = a + \frac{(s-4)}{\pi} -\!\!\!\!\!\!\int_4^\infty \frac{dx}{x-4}\frac{\omega(x)}{x-s}\,,$$

$$\omega(x) = \begin{cases} \sigma(x)^{-1}\sin^2\delta(x)\ ; & 4 \le x \le s_0 \\ A(x) & ;\ x \ge s_0, \end{cases} \tag{2.7}$$

with boundary condition (2.6). We shall refer to (2.1)–(2.4) or (2.6) and (2.7) as the *Roy equation with input* $(a, A)$. We assume in the following that this input is nonvanishing.

Once a solution of the Roy equation is known, the real part of the amplitude above $s_0$ is obtained from the dispersion relation (2.1), and $f$ is then defined on $[4, \infty)$.

The above formulation of the Roy equation is used in the following section. There exists an equivalent approach, based on the fact just mentioned that the dispersion relation (2.1) can be extended to the half axis $[4, \infty)$. This implies that the amplitude $f$ is the boundary value of an analytic function, holomorphic in $\mathbb{C}\backslash[4, \infty)$. The Roy equation then amounts to the construction of this analytic function. This method is described in detail in Sect. 4 and used in Sects. 5 and 6.

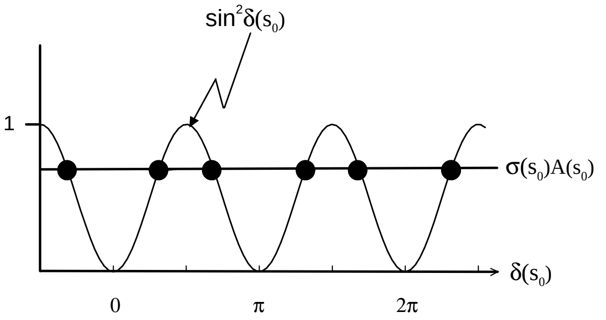

**Fig. 1.** Solutions to (2.6). The filled circles correspond to values of the phase shift $\delta(s_0)$ that fulfil the condition (2.6)

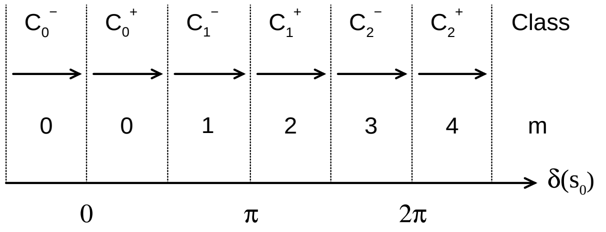

**Fig. 2.** The classes $C_n^\pm$ as defined in (2.9). The number $m$ in the lower row equals the dimension of the infinitesimal neighborhood of a given solution; see Sect. 3

The value of the phase shift at the matching point plays a crucial role in the following analysis [9,10,12]. However, the input absorptive part fixes $\delta(s_0)$ through the boundary condition (2.6) only modulo $\pi$ and up to its sign. In Fig. 1, we display $\sin^2\delta(s_0)$ as a function of the phase shift $\delta(s_0)$, together with the quantity $\sigma(s_0)A(s_0)$, shown with a horizontal line. The filled circles correspond to values of $\delta(s_0)$ that fulfil the condition (2.6). For reasons outlined in [10], we stick to phase shifts that are not too negative. Furthermore, the case where the phase shift at the matching point is a multiple of $\pi/2$ requires special consideration. This complication may be avoided with an appropriate choice of the matching point $s_0$ in actual calculations. Unless stated otherwise, we therefore assume in the following that

$$\begin{aligned} &\delta(s_0) > -\frac{\pi}{2}, \\ &\delta(s_0) \neq n\pi/2,\ n = 0, 1, 2, \ldots. \end{aligned} \tag{2.8}$$

The solution of the Roy equation is not unique in general [9,10,12], and it is useful to divide the manifold of solutions into classes $C_n^\pm$, parametrized by the value of the phase shift at the matching point,

$$\begin{aligned} C_n^+ &: \ \delta(s_0) \in \left(n\pi, n\pi + \frac{\pi}{2}\right), \\ C_n^- &: \ \delta(s_0) \in \left(n\pi - \frac{\pi}{2}, n\pi\right) \quad ; \quad n = 0, 1, 2, \ldots \end{aligned} \tag{2.9}$$

These classes are indicated in Fig. 2. To quote an example, a solution $f$ with $\delta(s_0) = 5\pi/4$ belongs to the class $C_1^+$.

Aside from the question of uniqueness, we will be concerned with the singularity structure of the solutions: every solution of (2.7) with arbitrary input $(a, A)$ is regular on $(4, s_0)$, but singular and only Hölder continuous at the end points of that interval. Whereas the singularity at $s = 4$ is due to the threshold behavior, the one at the matching point is unphysical, because the position of $s_0$ is arbitrary. It is only for the special class of analytic inputs – which will be specified in Sect. 6 – that there exists a solution that is regular at $s_0$.

Although the original system of coupled, nonlinear and singular integral equations has been reduced to the simplified equation (2.7), we are unable to solve it explicitly for an arbitrary input. For this reason, we have to rely on alternative tools to achieve our goal. In particular, we (i) investigate the infinitesimal neighborhood of a given solution, (ii) construct explicit solutions in the case where the matching point is moved to infinity, and (iii) investigate the manifold of solutions for a conveniently chosen specific input. A clear picture of the multiplicity of the solutions will emerge in this way, and the role of the unphysical singularity at the matching point can be investigated in a satisfactory manner.

## 3 Linearization and existence of ambiguities

We assume in this section that the Roy equation with input $(a, A)$ does have a solution $\delta$, and look for solutions $\delta'$ that are nearby. These can be determined by linearizing and explicitly solving the integral equation for the difference of the phase shifts. As already mentioned in the introduction, this method has two apparent drawbacks: first, one does not prove the very existence of the solution $\delta$. Second, it is not shown that the $m$-dimensional neighborhood of a solution is embedded in an $m$-dimensional manifold. On the other hand, as we will show in Sects. 4 and 5, we can construct explicit pairs $(a, A)$ for which a solution is known and for which the ambiguities found below are present. In any case we believe that, despite its shortcomings, the linearization method is very useful and illuminating, in particular so in view of its simplicity.

The solutions of the linearized integral equation show that, if a class with $n > 0$ is non-empty, it contains a continuous family of solutions [9]. This multiplicity structure is identical to the one indicated in the work of Atkinson and Warnock [10], and the investigations of the nonlinear case carried out in later sections support this picture.

We now show how this result is obtained when using the method of [9,12]. By assumption, the two phase shifts $\delta$ and $\delta'$ satisfy the integral equation (2.7). We wish to determine the difference

$$\Delta(s) = \delta'(s) - \delta(s). \tag{3.1}$$

Equation (2.7) for $\delta$ and $\delta'$ results in a nonlinear singular integral equation for $\Delta$. Assuming $\Delta$ to be small, linearization of this integral equation gives in the interval $s \in [4, s_0]$

$$\cos(2\delta(s))h(s) = (s-4)\frac{1}{\pi}\,⨍_4^{s_0} \mathrm{d}x \frac{1}{x-4}\frac{\sin(2\delta(x))h(x)}{x-s}\,, \tag{3.2}$$

where

$$h(s) = \frac{\Delta(s)}{\sigma(s)}. \tag{3.3}$$

To be consistent with the condition (2.6), we require furthermore that $\delta'(s_0) = \delta(s_0)$, or

$$h(s_0) = 0. \tag{3.4}$$

The original Roy equation is replaced by the singular linear integral equation (3.2), to be solved with the boundary condition (3.4). The latter shows that we can determine in this manner only those $f'$ which belong to the same class as $f$.

Constructing the solution of (3.2) is equivalent to solving a boundary value problem for analytic functions – a so-called Hilbert problem [27]. To make this article self-contained, we discuss the procedure in Appendix A, where it is shown that the general solution of (3.2) is given [9] by

$$h(s) = (s-4)G(s)P(s), \qquad s \in [4, s_0], \tag{3.5}$$

with

$$G(s) = \frac{1}{(s_0 - s)^m} \exp\left[\frac{2}{\pi}\,⨍_4^{s_0} dx \frac{\delta(x)}{x-s}\right], \tag{3.6}$$

and where $P(s)$ is an arbitrary real polynomial of degree $m - 1$, with

$$m = \begin{cases} \left[\frac{2\delta(s_0)}{\pi}\right] & \text{if } \delta(s_0) > \frac{\pi}{2}, \\ 0 & \text{if } -\frac{\pi}{2} < \delta(s_0) < \frac{\pi}{2}. \end{cases} \tag{3.7}$$

The symbol $[x]$ in (3.7) denotes the greatest integer not exceeding $x$. Polynomials of negative degree are considered to be identically zero here and in the following. For $\delta(s_0) > 0$, the number $m$ coincides with the so-called *index* of the Hilbert problem. For a monotonically increasing phase, it counts the number of times $\delta(s)$ goes through multiples of $\pi/2$ as $s$ varies from threshold to the matching point $s_0$. We indicate in Fig. 2 some of its values.

The following proposition summarizes the results obtained in this section and in Appendix A.

**Proposition 1** *Let $\delta$ be a solution of (2.7). It is an isolated solution of that equation if $-\pi/2 < \delta(s_0) < \pi/2$. If $\delta(s_0) > \pi/2$, the infinitesimal neighborhood of $\delta$ is an $m$-parameter family of solutions $\delta'$ with $\delta'(s_0) = \delta(s_0)$, where $m$ is given in (3.7).*

### Comments

1. An obvious question concerns the interpretation of the $m$ parameters in the polynomial $P$ in (3.5). In the case where the phase shift $\delta$ is monotonically increasing and where $m$ is an even integer, $f$ exhibits $m/2$ resonances on $(4, s_0)$. The nearby $f'$ has also $m/2$ resonances, with slightly modified positions and widths. These changes are fixed by the $m$ coefficients in $P$. This is verified in an example that we study in Subsect. 5.1. The situation is not so simple if $m$ is odd. The case $m = 1$ is illustrated in Subsect. 5.3.
2. Atkinson and Warnock [10] have studied these parameters in a single-channel Roy equation that has a structure which is similar to the one studied here. Using an $N/D$-method, they find that, when $m$ is an even integer, the parameters are related to the positions and

residues of the CDD poles [28] between threshold and the matching point. In the case where $m$ is an odd integer, one of these parameters is connected with the singular nature of the $N$ equation.

3. The difference $\triangle$ is singular at the matching point. This is a signal of the singularity of the general solution of the Roy equation mentioned in Sect. 2.
4. Our discussion deals exclusively with restrictions to the low-energy interval $[4, s_0]$. Once $\Delta$ is determined on that interval, the value of $f'$ for all real $s$ is obtained from the dispersion relation (2.1). In the linear regime,

$$f'(s) = f(s) + \frac{(s-4)}{\pi} \times \lim_{\epsilon \searrow 0} \int_4^{s_0} \mathrm{d}x \frac{\sin(2\delta(x))h(x)}{x-4} \frac{1}{x-s-\mathrm{i}\epsilon}. \quad (3.8)$$

Consistency with unitarity is then by no means ensured above $s_0$: if $\sigma f$ stays on or inside the Argand circle, $\sigma f'$ may well be outside. This shows that physical requirements which are not encoded in the Roy equation can reduce the ambiguities.

5. Our method is also applicable if the first inelastic threshold is below $s_0$, provided that the absorption parameter $\eta$ is known. The representation (2.5) is replaced by

$$f(s) = \frac{1}{2\mathrm{i}\sigma(s)} \left[\eta(s)\mathrm{e}^{2\mathrm{i}\delta(s)} - 1\right]. \quad (3.9)$$

The form of the linearized equation (3.2) is unchanged, whereas $\eta$ enters into the definition of the unknown, $h(s) = \eta(s)\sigma(s)^{-1}\Delta(s)$. With this modification, (3.5) remains valid.

## 4 The full amplitudes

### 4.1 Matching at infinity

It is instructive to consider the case where the matching point $s_0$ is moved to infinity, because the Roy equation can then be solved explicitly. The amplitude vanishes at infinity [17] in this case, as a result of which the input reduces to the scattering length $a$. Equation (2.7) becomes[2]

$$\frac{1}{2\sigma(s)} \sin(2\delta(s)) = a + \frac{(s-4)}{\pi} \fint_4^{\infty} \frac{\mathrm{d}x}{x-4} \frac{\sin^2 \delta(x)}{\sigma(x)(x-s)}, \quad s \in [4, \infty], \quad (4.1)$$

to be solved for given scattering length $a$. For $a > 0$, a solution is provided by the phase shift of the amplitude

$$f_1(s) = \left[\frac{1}{a} - \rho(s)\right]^{-1}, \quad (4.2)$$

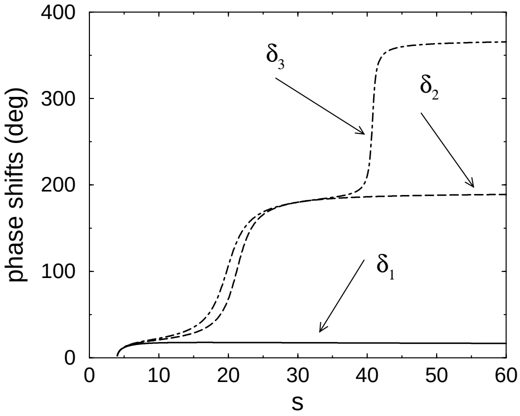

**Fig. 3.** The phase shifts $\delta_{1,2,3}$ that correspond to the solutions $f_{1,2,3}$ of (4.1), with parameters $a = 0.5$, $(s_2, r_2) = (30, 40), (s_3, r_3) = (45, 40)$

where $\rho$ is the Chew–Mandelstam function

$$\rho(s) = \frac{1}{\pi}\sigma(s) \left\{ \ln \frac{1-\sigma(s)}{1+\sigma(s)} + \mathrm{i}\pi \right\}; \quad s \geq 4. \quad (4.3)$$

In Fig. 3, we display the phase shift $\delta_1$ of $f_1$ with a solid line for $a = 0.5$. This is not the only solution – further examples are e.g. the phase shifts of

$$\frac{1}{f_2} = \frac{1}{f_1} + \frac{s-4}{s_2-4} \frac{r_2}{s-s_2}, \qquad \frac{1}{f_3} = \frac{1}{f_2} + \frac{s-4}{s_3-4} \frac{r_3}{s-s_3}; \quad s_i > 4, \ r_i > 0. \quad (4.4)$$

The corresponding phase shifts are again displayed in Fig. 3, for $(s_2, r_2) = (30, 40)$; $(s_3, r_3) = (45, 40)$. We note that the phase shifts tend to multiples of $\pi$ at infinity, $\delta_i(\infty) = (i-1)\pi, i = 1, 2, 3$. [The complications with $\delta(s_0) = n\pi/2$, that we mentioned in Sect. 2, disappear when $s_0 = \infty$.]

These examples show that the Roy equation with matching point at infinity allows for many solutions. The poles at $s_i$ are CDD poles, and one has to specify the phase shift at infinity as well as the CDD parameters $r_i, s_i$ in order to completely pin down a solution. The unsubtracted version of (4.1) is discussed in detail in [17].

### 4.2 Matching at finite energy

We now present an approach that will allow us to construct exact solutions of the Roy equation with a finite matching point $s_0$. Although we cannot solve the Roy equation with an arbitrary input, we can construct explicit amplitudes $f$ that satisfy the dispersion relation (2.1) and verify elastic unitarity below some $s_0$. An amplitude with this property defines an input $a_f \doteq f(4)$, $A_f(s) \doteq \mathrm{Im}\, f(s), s \geq s_0$, and is itself a solution of the Roy equation with this input. This holds true e.g. for the amplitudes $f_i$ in (4.2) and (4.4). Our goal is to show that – given $f$ – one can construct other solutions $f' \neq f$, with

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[2] A subtraction is in fact not needed here [17]. We stick to the present formulation for an easier comparison with the Roy equation at a finite matching point

the same input. In this manner, we find that the Roy equation has in general solutions in different classes $C_n^\mu$. This cannot be seen in the linearization framework discussed in the last section, since there one has to assume that the phase shifts of the original and of the new solution coincide at the matching point, as a result of which the old and new solutions stay in the same class.

Let us describe the procedure in detail. Until now we worked with amplitudes that are complex functions of the real variable $s \geq 4$. As announced in Sect. 2, we now define our amplitudes as analytic functions in the complex $s$-plane, cut along the real axis for $s \geq 4$. The corresponding amplitudes verifying (2.1)–(2.4) are the boundary values $f_+$ of $f$, defined as

$$f_+(s) = \lim_{\epsilon \searrow 0} f(s + \mathrm{i}\epsilon) \ , \qquad s \in [4, \infty). \tag{4.5}$$

In particular, we consider the set of functions $f$ with the following properties:

(i) $f$ is holomorphic in $\mathbb{C}\backslash[4,\infty)$ and verifies the dispersion relation (2.1), written for $f_+$.

(ii) $f_+$ is elastic below the matching point,

$$f_+(s) = \frac{1}{\sigma(s)} \mathrm{e}^{\mathrm{i}\delta(s)} \sin\delta(s), \qquad s \in [4, s_0]. \tag{4.6}$$

(iii) $f_+$ satisfies the regularity requirements listed in Subsect. A.2.

Let $f$ satisfy (i)–(iii). Its boundary value $f_+$ is a solution of the Roy equation with input

$$\begin{aligned} a_f &\doteq f(4), \\ A_f(s) &\doteq \mathrm{Im}\ f_+(s),\ s \geq s_0. \end{aligned} \tag{4.7}$$

In this and the following section, we show how to construct functions $f' \neq f$ that satisfy (i)–(iii) with

$$f'(4) = a_f, \tag{4.8a}$$

$$\mathrm{Im}\ f'_+(s) = A_f(s),\ s \geq s_0. \tag{4.8b}$$

It is clear that $f'_+$ is then also a solution of the Roy equation with input (4.7), and the existence of an $f'$ with the above-mentioned properties therefore establishes the non-uniqueness of the solution of the Roy equation.

In order to construct an $f'$, we use the following Ansatz:

$$\frac{1}{f'(s)} = \frac{1}{f(s)} + (s-4)\frac{H(s)}{D(s)}. \tag{4.9}$$

Here, $H$ is an Omnès-type function [9], an analogue of $\bar{G}$ defined in (A.9),

$$H(s) = \left(\frac{s_0}{s - s_0}\right)^m \exp\left[-\frac{2}{\pi} s \int_{s_0}^{\infty} \frac{\mathrm{d}x}{x} \frac{\theta(x)}{x - s}\right], \tag{4.10}$$

with $m$ given in (3.7). The function $\theta$ is Hölder continuous and equal to $\arg f_+$ modulo $\pi$, with $\theta(s_0) = \delta(s_0)$. (We assume here in addition that $\mathrm{Im}\ f_+ \geq 0$. As a result of this, we can define $\arg f_+$ such that $0 \leq \arg f_+ \leq \pi$.) The function $D$ is meromorphic[3] in $\mathbb{C}\backslash[s_0,\infty)$ – which ensures that (ii) is fulfilled – and has to be constructed such that also the remaining conditions are satisfied. We write

$$D = D_1 + D_2, \tag{4.11}$$

where $D_1$ is a meromorphic component of $D$, and where $D_2$ is regular in $\mathbb{C}\backslash[s_0,\infty)$. Condition (4.8a) requires $D(4) \neq 0$. The definition of $H$ has been chosen such that the condition (4.8b) amounts to a simple linear constraint on $D_2$,

$$\begin{aligned} \mathrm{Im}\ D_{2+}(s) &= \mu(s),\ s \geq s_0, \\ \mu(s) &= (s-4)|H(s)|A_f(s). \end{aligned} \tag{4.12}$$

Once this condition is fulfilled and $D(s_0)$ is not zero, $\mathrm{Im}\ f'$ is continuous at $s_0$ if $H(s_0) = 0$. This holds true if $\delta(s_0) > 0$. Choosing a particular function $D_2$ verifying the condition (4.12), the arbitrariness in $f'$ is entirely contained in $D_1$. This function has to be such that $f'$ satisfies (i) and (iii). Therefore, $1/f'$ and $1/f'_+$ have to be nonzero on $\mathbb{C}\backslash[4,\infty)$ and on $[4,\infty)$, respectively. The non-uniqueness of the solution of the Roy equation is due to the very existence of such functions $D_1$ – we provide explicit examples in the following section. Our discussion leads to

**Proposition 2** *Let $f$ be an amplitude verifying conditions (i)–(iii), with* $\mathrm{Im}\ f_+ \geq 0$, *and $\delta(s_0) > 0$. Let $f' \neq f$ also verify (i)–(iii), together with the conditions (4.8). Then $f'$ can be written in the form (4.9)–(4.11), where $D_1$ is a meromorphic component of $D$, and $D_2$ is regular in $\mathbb{C}\backslash[s_0,\infty)$, with spectral function (4.12).*

In the $N/D$ approach [10], an important part of the non-uniqueness is due to arbitrariness in the CDD poles. Their relation with the function $D$ in (4.11) is explained in Appendix B.

## 5 Explicit exact solutions

Here, we illustrate the procedure described in Subsect. 4.2 with specific examples. An amplitude fulfilling the conditions (i)–(iii) of that subsection can be written as follows:

$$f(s) = \left[\frac{1}{a} + (s-4)\phi(s) - \rho(s)\right]^{-1}, \tag{5.1}$$

where $\phi$ is a suitable function that is meromorphic in the complex $s$-plane, cut along the real axis for $s \geq s_0$. The amplitudes $f_i$ in (4.2) and (4.4) have this form with rational $\phi$. To keep our calculations simple, we work with the resonant amplitude $f_2$ in (4.4). We drop indices and write

$$f(s) = \left[\frac{1}{a} + \frac{s-4}{s_\mathrm{p} - 4}\frac{r}{s - s_\mathrm{p}} - \rho(s)\right]^{-1}, \tag{5.2}$$

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[3] In order to simplify the presentation, we shall – without any further mention – use the fact that all our analytic functions satisfy $F(s) = \overline{F(\bar{s})}$

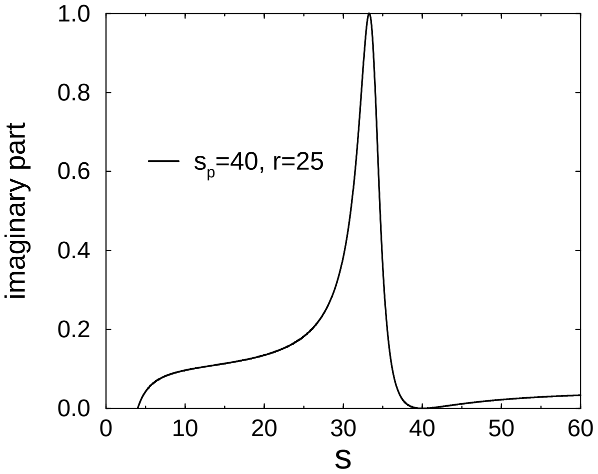

**Fig. 4.** The imaginary part of the function $\sigma f$ in (5.2). The parameters used are the ones in (5.3). The phase shift $\delta$ of $f$ is displayed with a solid line in Fig. 5

where $s_\mathrm{p} > 4$, $r > 0$, $a > 0$. As $f$ is elastic on $[4, \infty)$, the argument $\theta$ in (4.10) coincides with the phase shift $\delta$. In Figs. 4 and 5, we display the behavior of $f$ in the case where

$$a = 0.5, \quad s_\mathrm{p} = 40, \quad r = 25. \tag{5.3}$$

The imaginary part of $\sigma f$ is shown in Fig. 4, whereas its phase shift $\delta$ is indicated with a solid line in Fig. 5. Note that the matching point $s_0$ does not occur in $f$ – it may be chosen at one's convenience. We observe that $s_\mathrm{p}$ becomes a CDD pole – as defined in [10] – if $s_\mathrm{p} < s_0$.

In the following, we keep the amplitude $f$ fixed and bring it into class $C_1^+$ or $C_1^-$ by an appropriate choice of $s_0$. To illustrate, for $s_0 = 50$, $f$ is in class $C_1^+$, and $s_\mathrm{p}$ is a CDD pole.

We notice that $f$ is a solution of the Roy equation with input $(a_f, A_f)$ which is regular at $s_0$. We are thus dealing with one of the special inputs mentioned in Sect. 2.

### 5.1 Shift and suppression of a resonance: $f \in C_1^+$, $f' \in C_1^+, C_0^+$

We place $s_0$ above $s_\mathrm{p}$, as a result of which $\pi < \delta(s_0) < 3\pi/2$, such that $f$ belongs to class $C_1^+$. We first construct amplitudes $f'$ in the same class $C_1^+$. This implies that $\delta'(s_0) = \delta(s_0)$. We shall find that, qualitatively, $f'$ is obtained from $f$ by a shift of the position and a change of the width of its resonance. As $\delta'(s_0) \in (\pi, 3\pi/2)$, $1/f'$ has a single pole $s'_\mathrm{p}$ on $(4, s_0)$. The second term in the Ansatz (4.9) has to cancel the pole of $1/f$ at $s_\mathrm{p}$ and replace it by a new pole at $s'_\mathrm{p}$.

It is convenient to redefine $D_1$ and $D_2$ in (4.11) by writing

$$D(s) = (s - s_\mathrm{p})(s - s'_\mathrm{p})[D_1(s) + D_2(s)], \tag{5.4}$$

where $D_1$ is meromorphic, and

$$D_2(s) = \frac{1}{\pi} \int_{s_0}^{\infty} \mathrm{d}x \frac{\mu(x)}{(x - s_\mathrm{p})(x - s'_\mathrm{p})} \frac{1}{x - s}. \tag{5.5}$$

The integral converges because $\mu(x) = O(x)$ at infinity. We require regularity of $1/f'$ at $s_\mathrm{p}$ and fix the residue $r'$ of its pole at $s'_\mathrm{p}$, with

$$s'_\mathrm{p} \neq s_\mathrm{p}, \; 4 < s_\mathrm{p}, s'_\mathrm{p} < s_0, \; r' > 0. \tag{5.6}$$

This gives two conditions which completely determine a two-parameter Ansatz for $D_1$. One finds that the adequate Ansatz is

$$D_1(s) = \frac{1}{\alpha s + \beta}. \tag{5.7}$$

The two constraints on $D_1$ give

$$\alpha = \frac{\bar{r}'}{R'} - \frac{\bar{r}}{R}, \qquad \beta = \frac{\bar{r} s'_\mathrm{p}}{R} - \frac{\bar{r}' s_\mathrm{p}}{R'}, \tag{5.8}$$

with

$$\bar{r} = \frac{r}{(s_\mathrm{p} - 4) H(s_\mathrm{p})}, \qquad \bar{r}' = \frac{r'}{(s'_\mathrm{p} - 4) H(s'_\mathrm{p})},$$

$$R = 1 + \bar{r}(s_\mathrm{p} - s'_\mathrm{p}) D_2(s_\mathrm{p}), \; R' = 1 + \bar{r}'(s_\mathrm{p} - s'_\mathrm{p}) D_2(s'_\mathrm{p}). \tag{5.9}$$

The function $H$ is obtained from (4.10) with $m = 2$. Notice that $\alpha$ and $\beta$ are small if the pair $(s'_\mathrm{p}, r')$ is close to $(s_\mathrm{p}, r)$. In particular, let

$$\epsilon = \max\left( \frac{|s'_\mathrm{p} - s_\mathrm{p}|}{s_\mathrm{p}}, \; \frac{|r' - r|}{r} \right), \tag{5.10}$$

with $\epsilon$ small. We then have

$$\alpha = O(\epsilon), \qquad \beta = O(\epsilon). \tag{5.11}$$

Inserting the expressions (5.4) and (5.7) into (4.9) we get

$$\frac{1}{f'(s)} = \frac{1}{f(s)} + \frac{(s - 4) H(s)}{(s - s_\mathrm{p})(s - s'_\mathrm{p})} \frac{\alpha s + \beta}{1 + (\alpha s + \beta) D_2(s)}. \tag{5.12}$$

We see that $f'$ is regular in $\mathbb{C} \backslash [4, \infty)$ and is bounded on its cut if $1/f'$ has no zero. Equations (5.12) and (5.11) show that $1/f'$ is close to $1/f$ if $\epsilon$ is small and $s$ is outside the vicinity of $s_\mathrm{p}$ and $s'_\mathrm{p}$ and not too large. The inverse amplitude $1/f'$ is nonzero for such values of $s$ because $1/f$ is nonzero. One finds that $1/f'$ is also nonzero near $s_\mathrm{p}$ and $s'_\mathrm{p}$. Since $D_2$ and $H$ behave at infinity as $s^{-1} \ln |s|$ and $\ln^2 |s|$, respectively, the second term in (5.12) is no longer a small correction when $|s|$ becomes large, and a detailed analysis is needed. It is relatively easy to see that no unwanted zeros show up if $\epsilon$ is small and $\beta/\alpha$ is not too large. Consequently we are sure that under these conditions (5.12) provides a two-parameter family of solutions of the Roy equation. This example provides an illustration

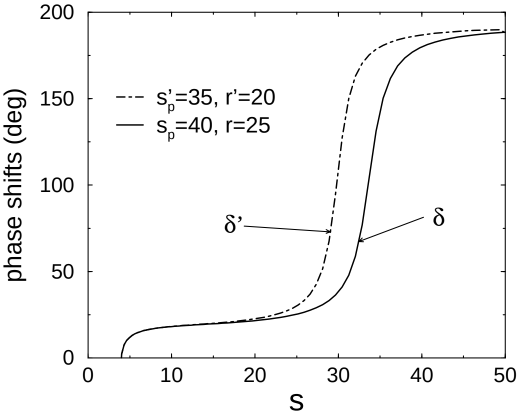

**Fig. 5.** A shift in the class $C_1^+$. The solid line corresponds to the phase shift $\delta$ of the function $f$ shown in Fig. 4, whereas the dot-dashed line displays the phase $\delta'$ of $f'$, evaluated from (5.12), with $\alpha, \beta$ calculated from (5.8) and (5.9), using the parameters (5.13)

of the $N/D$ framework outlined in Appendix B. In that language, $s_\mathrm{p}$ and $s'_\mathrm{p}$ correspond to zeros $z_j$. The CDD pole of $f$ at $s_\mathrm{p}$ is removed and replaced by a new one at $s'_\mathrm{p}$.

We have numerically verified that $\epsilon$ need not be small for the above conclusions to hold. In particular, there is a finite interval for $\epsilon$ such that (i) there are no additional singularities in $f'$, and (ii) the dispersion relation (2.1) is fulfilled. To illustrate this, we choose

$$s'_\mathrm{p} = 35,\ r' = 20,\ s_0 = 50. \tag{5.13}$$

In this case one has, using for $f$ the parameters displayed in (5.3),

$$\frac{|s'_\mathrm{p} - s_\mathrm{p}|}{s_\mathrm{p}} = 0.125,\ \frac{|r' - r|}{r} = 0.2. \tag{5.14}$$

These quantities are not small. Nevertheless, $f'$ is a solution of the Roy equation in class $C_1^+$. We display the phase shifts of $f$ and $f'$ in Fig. 5 with a solid and dot-dashed line, respectively. As required, the two phase shifts agree at the matching point $s_0$ – the parameters (5.13) really correspond to a shift $C_1^+ \to C_1^+$.

An expression for $f'$ can be obtained in the linear regime from the solution (3.5) of the problem on $[4, s_0]$. It coincides with the result (5.12) to first order in $\epsilon$, if the polynomial in (3.5) is

$$P(s) = C[(\bar{r}' - \bar{r})s + \bar{r}s'_\mathrm{p} - \bar{r}'s_\mathrm{p}], \tag{5.15}$$

where $C$ is a constant determined by the phase shift $\delta$. This establishes the relation with the coefficients in the arbitrary polynomial $P$ in our example and provides an interpretation of these coefficients.

The method allows the construction of solutions $f'$ which are no longer resonant. This is simply achieved by setting the residue $r'$ to zero. Equation (5.8) then gives $\alpha s + \beta = -\bar{r}(s - s'_\mathrm{p})/R$, and $1/f'$ becomes

$$\frac{1}{f'(s)} = \frac{1}{a} + \bar{r}\frac{s-4}{s-s_\mathrm{p}}\left[H(s_\mathrm{p}) - \frac{H(s)}{L(s)}\right] - \rho(s), \tag{5.16}$$

where

$$\begin{aligned} L(s) = 1 - \bar{r}\,[&(s - s_\mathrm{p})D_2(s) \\ &+ (s_\mathrm{p} - s'_\mathrm{p})\,(D_2(s) - D_2(s_\mathrm{p}))]\,. \end{aligned} \tag{5.17}$$

As $1/f'$ has no pole on $[4, s_0]$, $\delta'(s_0) = \delta(s_0) - \pi$, and $f' \in C_0^+$. We have to make sure that $1/f'$ has no unwanted zeros. The conclusions reached in the case $r' \neq 0$ apply here and $1/f'$ is non-vanishing if the residue $r$ is small, i.e. if the resonance in $f$ is narrow. On also finds that Re $(1/f')$ does not have a zero on $[4, s_0]$ either. Therefore, the phase shift $\delta'$ stays below $\pi/2$, and $f'$ is non-resonant. It is easy to see that $L$ does not depend on the choice of $s'_\mathrm{p}$. This means that the amplitude $f'$ defined in (5.16) effectively contains no free parameter. This amplitude is the unique solution of the Roy equation with input (4.8) belonging to class $C_0^+$. This uniqueness is in accordance with the results of Sect. 3. In the $N/D$ language, the uniqueness stems from the fact that $f'$ has no CDD pole, and $\delta'(s_0) < \pi/2$. What is new with respect to Sect. 3 is that our example shows explicitly that the same Roy equation has solutions in different classes $C_n^\pm$. At given input $(a, A)$, the number of resonances below $s_0$ is not fixed unless one imposes the precise value of $\delta'(s_0)$.

### 5.2 Implantation of a resonance: $f \in C_1^+$, $f' \in C_2^+$

We corroborate our last statement by examining another Ansatz for $D_1$ in the context of the last subsection. We replace (5.7) by

$$D_1(s) = \frac{1}{\alpha}(s - s_1)\,, \tag{5.18}$$

and again require that $1/f'$ obtained from (4.9) and (5.4) be regular at $s_\mathrm{p}$ and have a pole at $s'_\mathrm{p}$ with residue $r'$, with conditions (5.6). One finds

$$\begin{aligned} \alpha &= \frac{1}{N}\bar{r}\bar{r}'(s'_\mathrm{p} - s_\mathrm{p})^2, \\ s_1 &= \frac{1}{2}(s'_\mathrm{p} + s_\mathrm{p}) - \frac{1}{2N}(s'_\mathrm{p} - s_\mathrm{p}) \\ &\quad \times \left\{\bar{r}' + \bar{r} - \bar{r}'\bar{r}(D_2(s'_\mathrm{p}) + D_2(s_\mathrm{p}))(s'_\mathrm{p} - s_\mathrm{p})\right\}\ , \\ N &= \bar{r} - \bar{r}' + \bar{r}\bar{r}'(s_\mathrm{p} - s'_\mathrm{p})\left\{D_2(s'_\mathrm{p}) - D_2(s_\mathrm{p})\right\}. \end{aligned} \tag{5.19}$$

We see that in general $\alpha = O(\epsilon)$, whereas $s_1$ is anywhere on the real axis. Equation (5.12) is replaced by

$$\frac{1}{f'(s)} = \frac{1}{f(s)} + \frac{(s-4)H(s)}{(s-s_\mathrm{p})(s-s'_\mathrm{p})}\,\frac{\alpha}{s - s_1 + \alpha D_2(s)}. \tag{5.20}$$

If $\epsilon$ is small, $1/f'$ has a pole $\bar{s}_1$ near $s_1$ with small residue, $O(\epsilon)$. This implies that $1/f'$ exhibits a zero near $s_1$ which can preclude the Ansatz (5.18). The discussion becomes delicate if $s_1$ is close to 4 or $s_0$ and the next statements are valid if $\bar{s}_1$ is outside $O(\epsilon)$ neighborhoods of 4 and $s_0$:

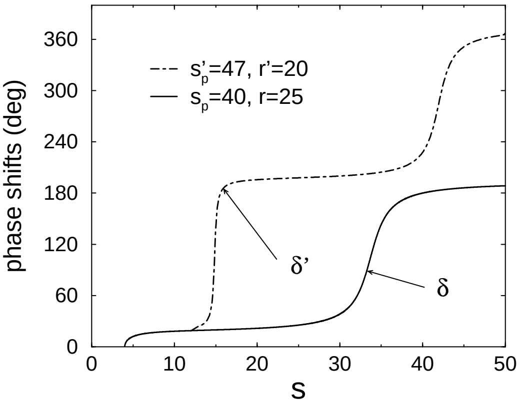

**Fig. 6.** The phase shifts corresponding to a shift $C_1^+ \to C_2^+$. The solid line is the phase shift $\delta$ of the function $f$ shown in Fig. 4, whereas the dot-dashed line displays the phase shift $\delta'$ of $f'$, evaluated from (5.20). The parameters $\alpha$ and $s_1$ are evaluated from (5.19), with (5.21)

(i) if $\bar{s}_1 < 4$ or $\bar{s}_1 > s_0$, $f'$ has a pole on the real axis near $\bar{s}_1$;

(ii) if $4 < \bar{s}_1 < s_0$, $f'$ has a pair of complex conjugate poles close to $\bar{s}_1$. These poles are in the first sheet of the branch point $s = 4$ if $\alpha < 0$; they are in the second sheet if $\alpha > 0$.

We see that the Ansatz (5.18) has to be rejected for pairs $(s'_{\rm p}, r')$ such that $\bar{s}_1 < 4$ or $\bar{s}_1 > s_0$. It is also inadequate if $4 < \bar{s}_1 < s_0$ and $\alpha < 0$. These requirements on $\alpha$ and $\bar{s}_1$ are fulfilled in a wedge-shaped domain of the $(s'_{\rm p}, r')$-plane, with apex at $(s_{\rm p}, r)$.

If $\alpha > 0$, $f'$ generates a solution of the Roy equation in class $C_2^+$ ($\delta'(s_0) = \delta(s_0) + \pi$). This solution displays two resonances on $[4, s_0]$: a shifted resonance, $(r, s_{\rm p}) \to (r', s'_{\rm p})$, and an implanted narrow resonance near $s_1$. We illustrate this transition in Fig. 6. There, we display the phase shifts of $f$ and $f'$ with a solid and a dot-dashed line, respectively, for

$$s'_{\rm p} = 47,\ r' = 20,\ s_0 = 50. \tag{5.21}$$

These parameters are again outside the linear regime. The difference of the phase shifts is exactly $\pi$ at the matching point $s_0 = 50$. This amounts to a special case where the CDD pole at $s_{\rm p}$ is replaced by two poles at $s'_{\rm p}$ and $\bar{s}_1$.

We have pointed out in Sect. 3 that the new solutions exhibit a singular behavior at the matching point $s_0$. We illustrate this feature in Fig. 7, where we display the real and the imaginary part of $f$ and of $f'$ with a solid and a dot-dashed line, respectively, for the situation displayed in Fig. 6. The singular behavior of $f'$ is manifest, both in the real and in the imaginary part. Note that, since the imaginary parts of $f$ and $f'$ agree for $s \geq s_0$, the real parts also agree at $s = s_0$ due to unitarity. Above the matching point, they are, however, in general different. Because the real part of $f$ is positive in the present case, the figure shows that $f'$ is inside the Argand diagram, at least in an interval above $s_0$.

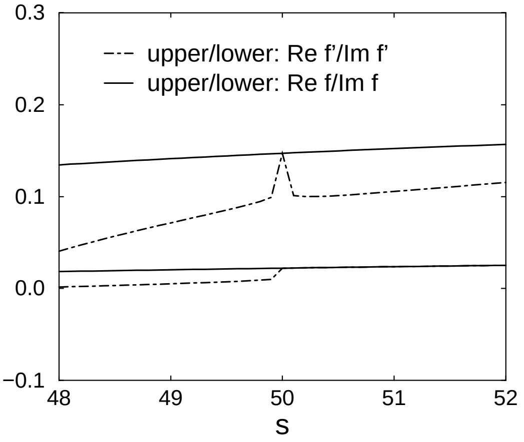

**Fig. 7.** The cusps in $f'$. We display the real and imaginary parts of $f$ and $f'$, for the situation displayed in Fig. 6

### 5.3 Shift of a resonance: $f, f' \in C_1^-$

In this example we bring $f$ in (5.2) into $C_1^-$ by pushing $s_0$ below $s_{\rm p}$ in such a manner that the resonance position is still below $s_0$, i.e. $\pi/2 < \delta(s_0) < \pi$. [The pole at $s_{\rm p}$ is no longer a CDD pole.] The function $H$ used in the previous two subsections now has to be replaced by a new one that we call $\hat{H}$. It is given by (4.10), with $m = 1$. Correspondingly, $\mu$ defined in (4.12) becomes $\hat{\mu}$. It behaves as $s^2$ at infinity and three subtractions are needed in the construction of $D_2$. A convenient redefinition of $D_{1,2}$ in (4.11) leads to

$$D(s) = (s - s_{\rm p})\left[D_1(s) + (s - s_{\rm p})D_2(s)\right], \tag{5.22}$$

where

$$D_2(s) = s\frac{1}{\pi}\int_{s_0}^{\infty}\frac{{\rm d}x}{x}\frac{\hat{\mu}(x)}{(x - s_{\rm p})^2}\frac{1}{x - s}. \tag{5.23}$$

The integral converges because $\hat{\mu}$ has a second-order zero at $s_{\rm p}$. As before, we require $1/f'$ to be regular at $s_{\rm p}$. This implies

$$\hat{r} + \frac{\hat{H}(s_{\rm p})}{D_1(s_{\rm p})} = 0, \tag{5.24}$$

with $\hat{r} = r/(s_{\rm p} - 4)$. This equation is consistent with real values of $\hat{r}$ and $D_1(s_{\rm p})$, because $\hat{H}(s_{\rm p})$ is real. A suitable Ansatz for $D_1$ is

$$D_1(s) = -\frac{\hat{H}(s_{\rm p})}{\hat{r}(s_{\rm p} - s'_{\rm p})}(s - s'_{\rm p}). \tag{5.25}$$

Condition (5.24) is fulfilled with $s'_{\rm p}$ a free parameter, $s'_{\rm p} > s_0$. Insertion of (5.24) and (5.25) into (4.9) gives

$$\frac{1}{f'(s)} = \frac{1}{a} + (s - 4)\hat{r}\left\{\left[1 - \frac{\hat{H}(s)}{\hat{H}(s_{\rm p})}\right]\frac{1}{s - s_{\rm p}} + \frac{\hat{H}(s)}{\hat{H}(s_{\rm p})}\frac{1}{s - s'_{\rm p} + L(s)}\right\} - \rho(s), \tag{5.26}$$

where

$$L(s) = \bar{r}\frac{(s'_{\rm p} - s_{\rm p})^2 D_2(s)}{1 + \bar{r}(s'_{\rm p} - s_{\rm p})D_2(s)}. \qquad (5.27)$$

Using $\epsilon$ defined in (5.10), we have $L = O(\epsilon^2)$ as long as $s$ is not too large. For such values of $s$, $1/f'$ is close to $1/f$ outside a neighborhood of $s_{\rm p}$ and $s'_{\rm p}$. We notice that $\mathrm{Im}\, L_+ \neq 0$ for $s > s_0$ and $1/f'$ has no real pole whose real residue $r'$ could be imposed (there is a pair of complex conjugate poles near $s'_{\rm p}$ located in higher sheets of the logarithmic branch point at $s_0$). One finds that $1/f'$ has no unwanted zeros for any $s$ if $\epsilon$ is small enough, and we end up with a family of solutions of the Roy equation in $C_1^-$ indexed by the single parameter $s'_{\rm p}$. This is in accordance with Sect. 3. The shift of the resonance position and the change of its width are now correlated and fixed by the value of $(s'_{\rm p} - s_{\rm p})$. Both $f$ and $f'$ are without CDD poles, and the non-uniqueness is due to the fact that $\pi/2 < \delta(s_0) < \pi$.

## 6 How to ensure uniqueness

The solution of the Roy equation is not unique – the behavior of a partial wave below $s_0$ cannot be predicted in a unique way from an input $(a, A)$. Information on the value of the phase shift at $s_0$ is needed and free parameters have to be fixed if $\delta(s_0) > \pi/2$. This non-uniqueness severely restricts the efficiency of the Roy equation as a tool for the construction of a low-energy extrapolation. We show in the following how uniqueness can be restored in principle by imposing additional physical requirements. The unphysical singularity at the matching point is removed at the same time, as it has to be.

### 6.1 Examples of unique solutions

We start with three illustrative examples.

#### 6.1.1 Example 1

In Sect. 5 we have constructed, starting from the amplitude (5.2), several functions $f'$ that satisfy the Roy equation with matching point $s_0 = 50$ and input generated by $f$,

$$a = f(4); \quad A(s) = \mathrm{Im}\, f(s),\ s \geq s_0\,. \qquad (6.1)$$

Two cases are displayed – in terms of their phase shifts $\delta'$ – in Figs. 5 and 6. As we already mentioned, these phase shifts develop singularities at the matching point, see Fig. 7. Below we will show that, had we required the new solution $f'$ to be regular at $s_0$, there would be exactly one solution of the Roy equation with input (6.1), namely $f$ itself. In other words, all solutions $f' \neq f$ develop singularities at $s_0$. It turns out that this property of the input (6.1) is due to the fact that the amplitude is elastic above the matching point. Whereas $f$ is elastic on the whole interval $[4, \infty)$, the following example shows that a finite interval containing $[4, s_0]$ suffices to render the solution unique.

#### 6.1.2 Example 2

We consider the function $f'$ constructed in Subsect. 5.2, see (5.20), (5.21) and Fig. 6, dot-dashed line. Suppose we wish to construct solutions of the Roy equation with displaced matching point $s'_0 = 45$ and input defined by the scattering length and the absorptive part of $f'$,

$$a = f'(4)\,;\ A(s) = \mathrm{Im}\ f'(s)\,, s \geq s'_0\,. \qquad (6.2)$$

In the language of Sect. 2, this problem belongs to the class $C_2^-$: the phase shift at the matching point is $3\pi/2 < \delta'(s'_0) < 2\pi$, see Fig. 6, and according to proposition 1 in Sect. 3, the Roy equation with input (6.2) has therefore a three-dimensional manifold of solutions. The uniqueness statement in Example 1 is also true here: Suppose we seek for solutions of the Roy equation with input (6.2) and require that the solution is regular at $s = s'_0$. There is again exactly one solution, namely $f'$ itself.

It is obvious that we are dealing with special inputs – we call them *analytic inputs* below. The following example displays an input that is not analytic.

#### 6.1.3 Example 3

Consider again the amplitude (5.2). We set the residue $r$ to zero,

$$a = f(4) + \epsilon\,;\ A(s) = f(s)\,,\ s \geq s_0\,;\ r = 0\,, \qquad (6.3)$$

with $s_0 = 50$. For sufficiently small $\epsilon \neq 0$, one can show that there is *no* solution that is analytic at $s_0$; see Subsect. 6.3.

### 6.2 Analytic input and uniqueness

To generalize our findings, we exploit smoothness properties of the amplitudes which are not explicit in the Roy equations. The partial-wave amplitudes enter this equation as boundary values of analytic functions. Boundary values need not be smooth, and this is compatible with the dispersion relation (2.1). However, it is quite remarkable that smoothness is imposed by elastic unitarity:

**Proposition 3** *Let $f$ be regular in the complex $s$-plane, cut along the real axis for $s \geq 4$, and let its boundary value $f_+$ verify elastic unitarity on* $[4, s_1]$. *Then the real and imaginary parts of $f_+$ are separately holomorphic in a complex neighborhood of* $(4, s_1)$.

A proof of the proposition is given in [26]. Notice that the parameter $s_1$ need not coincide with the first inelastic threshold $s_{\rm inel}$ – the proposition is true for any $s_1 > 4$. Taking $s_1 = s_0$, one concludes that all solutions $\delta$ of the Roy equation (2.7) are regular in the interval $(4, s_0)$.

Consider amplitudes that satisfy the conditions of proposition 3 and in addition verify the Roy equation with input $(a_f, A_f(s)) = (f(4), \mathrm{Im}\ f_+(s))$ for some $s_0 < s_1$. The proposition tells us that $f$ is regular at $s_0$. A second solution of the same equation may be singular at $s_0$, because it need not be elastic on $[s_0, s_1]$. The following proposition shows that $f$ is in fact the only solution which is regular at $s_0$. This is an important feature, allowing one in principle to identify the physical extrapolation within the manifold of solutions.

**Proposition 4** *Let $f$ be an amplitude that satisfies the conditions of proposition 3 and that furthermore verifies the Roy equation with input $(a_f, A_f)$ for some $s_0 < s_1$. Let $f' \neq f$ be a second solution with the same input. Then $f$ is regular at $s_0$, whereas $f'$ is singular.*

We relegate a proof of this proposition to Appendix C. It elaborates on observations made in [9] and [12]. Examples 1 and 2 given above fulfill the conditions of the proposition with $s_1 = \infty$ and $s_1 = 50$, respectively. Therefore, according to proposition 4, there is exactly one solution of the Roy equation that is regular at the matching point.

Proposition 4 tells us that there is a special class of inputs that allow a unique solution that is regular at the matching point $s_0$, as announced in Sect. 2. According to proposition 3, the high-energy absorptive part of an input belonging to that class has an analytic continuation from $[s_0, s_1)$ into a complex neighborhood of $(4, s_1)$. For this reason, we say that the members of our special class are *analytic inputs*. A physical amplitude $f$ defines an analytic input $(a_f, A_f)$ if $s_0 < s_{\text{inel}}$. Uniqueness is achieved in the sense that the corresponding Roy equation has exactly one solution, coinciding with $f$, which is regular at $s_0$.

The existence of a class of inputs ensuring uniqueness and regularity at the matching point has been established in an indirect way. We have no direct and complete characterization of an analytic input. There are involved constraints apart from analyticity of the high-energy absorptive part $A$, and one cannot decide directly if a given input is an analytic one. In particular, the scattering length $a$ is fixed by $A$ and is not an independent parameter; see Example 3 above. We arrived at a unique solution of the Roy equation by choosing inputs which are compatible with elastic unitarity above the matching point. In the physical context, this means $s_{\text{inel}} > s_0$. One can prove that uniqueness is also obtained if $s_0 > s_{\text{inel}}$, provided that the inelasticity is sufficiently smooth.

### 6.3 Approximate input and cusps

Although we are working here with model amplitudes without left-hand cut, the results of Subsect. 6.2 hold true in the physically realistic situation with left-hand cut. As a physically relevant input is analytic, with $s_1 = s_{\text{inel}}$, we conclude that it allows a unique solution regular at $s_0$ if $s_0 < s_{\text{inel}}$. However, a physical input is only approximately known, and one is faced in practice with an arbitrary input, as a result of which non-uniqueness and unphysical singularities do occur. This is the reason why we have analyzed in detail the Roy equation in its general setting.

If we know that a high-energy absorptive part $A$ belongs to an analytic input, but the corresponding scattering length is not precisely known, one has to work with a trial input $(a', A)$, with $a' \neq a$. We expect that all the resulting solutions will have cusps at $s_0$. This can be established if $a'$ is infinitesimally close to $a$ by using the techniques developed in [9,12]. For instance, if $\delta(s_0) < \pi/2$ and $\delta'(s_0) = \delta(s_0)$, the neighboring solution $f'$ is unique and its phase shift is given by

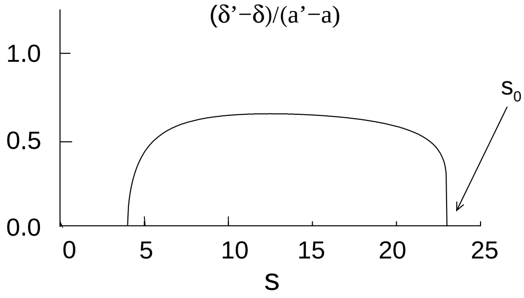

**Fig. 8.** The quantity $(\delta' - \delta)/(a' - a)$ according to (6.4). The reference amplitude $f$ is the one in (5.2), and the parameters used are given in (6.5). The cusp generated at $s_0$ is very clearly seen

$$\delta'(s) = \delta(s) + \sigma(s)\frac{G(s)}{G(4)}(a' - a)\,, \tag{6.4}$$

with $G$ defined in (3.6). The ratio $(\delta' - \delta)/(a' - a)$ is shown in Fig. 8 for a neighboring solution of the amplitude $f$ in (5.2), using

$$s_{\text{p}} = 40,\ r = 100,\ s_0 = 23. \tag{6.5}$$

The cusp at $s_0$ is clearly visible and the effect of a modified scattering length extends over the whole interval $(4, s_0)$. If an incorrect value $a'$ of the scattering length also entails a cusp when it is not close to $a$, the correct scattering length is specified by the fact that it defines an input allowing a unique solution without a cusp: one solves the Roy equation with several trial $a'$, and uses

$$a \doteq a'_{nocusp}\,. \tag{6.6}$$

In this sense the scattering length of a physically realistic amplitude can be predicted in principle. However, even if we are sure that the high-energy absorptive part is analytic, we do not know it exactly in practice, and we have to use an approximate form $A'$. We have to work with an input $(a', A')$ which is meant to approximate the analytic $(a, A)$. It is unlikely that there will still be a value of $a'$ which removes the cusp in one of the solutions of the resulting Roy equation. What one can try in practice is to find the value of $a'$ which minimizes the size of the cusp. In the coupled channel case, an analogous procedure may be used to avoid solutions that generate a cusp [4].

## 7 Summary and conclusions

The following points summarize the content of this article.

1. In view of forthcoming applications [4] of the Roy equations [3] in the analysis of $K_{l4}$ decays, we have considered here the one-channel Roy equation. We have analyzed the multiplicity and singularity structure of its solutions for a given input $(a, A)$ of scattering length $a$ and high-energy absorptive part $A$.
2. First, we have investigated the infinitesimal neighborhood of a given solution $\delta$ [9]. According to proposition 1 in Sect. 3, this neighborhood contains an $m$-parameter family of solutions, where
$$m = \begin{cases} \left[\frac{2\delta(s_0)}{\pi}\right] & \text{if } \delta(s_0) > \frac{\pi}{2}, \\ 0 & \text{if } -\frac{\pi}{2} < \delta(s_0) < \frac{\pi}{2}. \end{cases} \tag{7.1}$$
The symbol $[x]$ denotes the greatest integer not exceeding $x$. For a monotonically increasing phase, $m$ counts the number of times $\delta(s)$ goes through multiples of $\pi/2$ as $s$ varies from threshold to the matching point $s_0$. This result illustrates that a given input $(a, A)$ does not, in general, uniquely fix the solution. One has in addition to fix the phase shift at the matching point, and one has to determine the corresponding $m$ parameters by other means.
3. Using proposition 2 of Sect. 4, we have constructed in Sect. 5 – starting from a given solution $f$ with input $(a_f, A_f)$ – additional exact solutions $f' \neq f$ with the same input. The function $f'$ contains in general several arbitrary parameters that may be used to either change the position and residues of the poles present in the inverse amplitude $1/f$, or to remove (or implement new) poles. This illustrates that a given input allows for solutions with a different value of the phase shift at the matching point – these phase shifts only have to satisfy the boundary condition (2.6).
4. The solutions $f' \neq f$ so constructed have the property that they contain a cusp in the real and imaginary part at the matching point. An example is displayed in Fig. 7. The origin of these cusps is made clear in proposition 4 of Sect. 6.
5. In case we know that a given absorptive belongs to an analytic input (see Sect. 6 for this notion), we expect on the basis of proposition 4 that the corresponding scattering length can be determined as the one that results in a solution of the Roy equation without a cusp at $s_0$.
6. Propositions 1 and 3 in Sects. 3 and 6 were established long ago in [9] and [26], respectively. As far as we are aware, propositions 2 and 4 in Sects. 4 and 6 are, on the other hand, new results.

In conclusion, if we know that we are dealing with an analytic input, the corresponding amplitude is the unique regular solution of the Roy equation. Because we are in general forced to use an approximate input, non-uniqueness and unphysical singularities do show up. A visible cusp in a numerical solution of the Roy equation with $0 < \delta(s_0) < \pi/2$ is a signal that the input is not physical or a poor approximation of a physical one. The deficiency may be hidden in the scattering length $a$, in the absorptive part $A$, or in both.

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## A Solving the integral equation (3.2)

We first specify the regularity properties of the partial waves and phase shifts considered in the text. Then we construct the general solution of the integral equation (3.2).

### A.1 Hölder continuity

The class of Hölder-continuous functions is the appropriate space to consider, because Cauchy integrals map this space essentially into itself [27]. Let us explain this notion.

Consider a complex-valued function $f$ of a real variable $x$. The function $f$ is called Hölder continuous in the interval $[a, b]$ with exponent $\mu$, where $0 < \mu \leq 1$, if there is a constant $C$ such that
$$|f(x) - f(y)| \leq C\,|x - y|^{\mu} \quad ; \ x, y \in [a, b]. \tag{A.1}$$
We call these functions $H$-continuous for short, and denote the corresponding space with $H^{\mu}_{a,b}$.

**Proposition A** *Let $f \in H^{\mu}_{a,b}$, with $f(a) = f(b) = 0$ and $\mu < 1$. Then the function*
$$g(y) = \int_a^b \!\!\!\!\!\!- \ \mathrm{d}x \frac{f(x)}{x - y} \tag{A.2}$$
*is also an element of $H^{\mu}_{a,b}$, with the same exponent $\mu$. The same is true for*
$$g_{\pm}(y) = \lim_{\epsilon \searrow 0} \int_a^b \mathrm{d}x \frac{f(x)}{x - y \mp \mathrm{i}\epsilon}. \tag{A.3}$$
The proof may be found in [27].

### A.2 Regularity requirements

The input absorptive part $A$ in (2.1) is assumed to be bounded and Hölder continuous in any finite interval $[s_0, s_0']$ above the matching point $s_0$. Furthermore, we seek solutions of the Roy equation that are $H$-continuous in $[4, s_0]$ and have normal threshold behavior (which implies $\mu \leq 1/2$):
$$\begin{aligned} &\text{(i)} \quad A \in H^{\mu}_{s_0,s_0'}, \ s_0' > s_0; \quad A \ \text{is bounded}, \\ &\text{(ii)} \quad f \in H^{\mu}_{4,s_0}, \\ &\text{(iii)} \quad f = a + \mathrm{i}a^2\sigma(s) + O(s - 4), \ s \searrow 4. \end{aligned} \tag{A.4}$$

The phase shifts can then be chosen $H$-continuous as well,
$$\begin{aligned} &\text{(iv)} \quad \delta \in H^{\mu}_{4,s_0}, \\ &\text{(v)} \quad \delta = \sigma(s)[a + O(s - 4)], \ s \searrow 4. \end{aligned} \tag{A.5}$$

### A.3 Solution of the integral equation (3.2)

We first show that solving the integral equation (3.2) with the boundary condition (3.4) is equivalent [27] to solving a boundary value problem known as a *Hilbert problem.* We then construct the general solution of the latter.

We start by introducing an auxiliary function $\Phi$ [9,12],

$$\Phi(z) = (z-4)\frac{1}{\pi}\int_4^{s_0} \mathrm{d}x \frac{\sin(2\delta(x))h(x)}{x-4}\frac{1}{x-z}. \qquad (A.6)$$

For convenience, we use in this subsection the variable $z$ to indicate complex values of the variable $s$. The function $\Phi$ has the following properties if $h$ is a solution of (3.2) and (3.4):

(i) $\Phi(z) = \overline{\Phi(\bar{z})}$ is regular in $\mathbb{C}\backslash[4, s_0]$.
(ii) The boundary values $\Phi_\pm(s) = \lim_{\epsilon\searrow 0} \Phi(s \pm \mathrm{i}\epsilon)$ are $H$-continuous in $[4, s_0]$.
(iii) $\Phi_+(s) = \mathrm{e}^{4\mathrm{i}\delta(s)}\Phi_-(s); \quad s \in [4, s_0]$.
(iv) $\Phi(4) = 0$.
(v) $\Phi(s_0) = 0$.
(vi) $\Phi$ is bounded at infinity.

Out of these, we only prove property (ii) – the remaining ones are easy to verify. The unknown $h$ in (A.6) is an element of $H^\mu_{4,s_0}$, with $0 < \mu \leq 1/2$. This follows from (A.5) – the same equation shows that $h = O(s-4)$ at $s = 4$. Together with $h(s_0) = 0$ and with proposition A in Subsect. A.1, it follows that $\Phi_\pm(s) \in H^\mu_{4,s_0}$, with the same exponent $\mu$.

The problem to determine functions $\Phi(z)$ with (i)–(vi) is called a *(homogeneous) Hilbert problem.* We conclude that each solution of the original integral equation generates via (A.6) a solution of (i)–(vi). Vice versa, each solution of the Hilbert problem has a representation of the form (A.6), where $h$ is given by

$$h(s) = \mathrm{e}^{-2\mathrm{i}\delta(s)}\Phi_+(s). \qquad (A.7)$$

Furthermore, the function $h$ is real and solves the integral equation (3.2). To prove this, we write a Cauchy representation for $[\Phi(z) - \Phi(4)]/(z-4)$, with a path that wraps around the cut $[4, s_0]$. We then deform the outer part of the path towards infinity, in such a manner that only the integral above and below the cut $[4, s_0]$ survives. By use of condition (iv), we find

$$\Phi(z) = \frac{(z-4)}{2\pi\mathrm{i}}\int_4^{s_0} \frac{\mathrm{d}x}{x-4}\frac{\Phi_+(x) - \Phi_-(x)}{x-z}. \qquad (A.8)$$

Using properties (i) and (iii), the claim is easily proven, and we conclude that solving (3.2) with (3.4) indeed is equivalent to solving the Hilbert problem (i)–(vi).

It remains to construct the general solution of (i)–(vi). First, we observe that the Omnès-type function [9]

$$\bar{G}(z) = \frac{1}{(s_0-z)^m}\exp\left[\frac{2}{\pi}\int_4^{s_0} \mathrm{d}x\frac{\delta(x)}{x-z}\right], \qquad (A.9)$$

with $m$ defined in (3.7), satisfies property (i). The behavior at $s_0$ is given by

$$\bar{G}(z) \sim (s_0-z)^\gamma, \qquad \gamma = \frac{2}{\pi}\delta(s_0) - m, \qquad (A.10)$$

$-1 < \gamma < 1$. The $H$-continuity and threshold behavior of $\delta$ imply that $\bar{G}$ satisfies (ii) and (iii) except, possibly, at $s_0$. Outside $s_0$, $(s_0-z)^m\bar{G}$ and its boundary values are nonzero and the function

$$F(z) = (s_0-z)\frac{\Phi(z)}{\bar{G}(z)} \qquad (A.11)$$

is regular in $\mathbb{C}\backslash\{s_0\}$. For $z \neq s_0$ it is given by its Laurent series. The principal part of this series is identically zero because condition (v) and (A.10) imply that $F(s_0) = 0$. $F(z)$ is therefore an entire function. Condition (vi) tells us that $F(z) \sim z^{m+1}$ at infinity, as a result of which

$$F(z) = (s_0-z)Q(z), \qquad (A.12)$$

where $Q(z)$ is a polynomial of degree $m$. Condition (iv) imposes $Q(4) = 0$, and this gives $Q = 0$ if $m = 0$: the conditions (i)–(vi) allow only for the trivial solution in this case. If $m > 0$, $Q(z) = (z-4)P(z)$, with $P$ a polynomial of degree $m-1$. We conclude that the general solution of (i)–(vi) is given by

$$\Phi(z) = (z-4)\bar{G}(z)\sum_{n=0}^{m-1} c_n z^n, \; c_n \in \mathbb{R}, \qquad (A.13)$$

with $\Phi = 0$ for $m = 0$. Using (A.7), the result (3.5) follows.

## B Connection with the $N/D$ approach

The $N/D$ method [10] transforms the nonlinear Roy equation into a linear $N$ equation, whereas the method displayed in Sect. 4 linearizes the construction of the solution $f'$, once a first solution $f$ is known. In this Appendix, we establish the relationship between the two methods.

We write the $N/D$ representation of $f$ as follows:

$$f(s) = \frac{n(s)}{d(s)}. \qquad (B.1)$$

The $N$-function $n$ is holomorphic in $\mathbb{C}\backslash[s_0, \infty)$, and the $D$-function $d$ is holomorphic in $\mathbb{C}\backslash[4, s_0]$, with possible CDD poles on the cut. Similarly, $f' = n'/d'$. We now determine the relation between the pairs $(n, d)$ and $(n', d')$.

Equation (4.9) gives

$$f'(s) = \frac{D(s)}{F(s)}f(s), \qquad (B.2)$$

where

$$F(s) = D(s) + (s-4)H(s)f(s). \qquad (B.3)$$

Whereas $D$ is meromorphic in $\mathbb{C}\backslash[s_0, \infty)$, $F$ is meromorphic in $\mathbb{C}\backslash[4, s_0]$. This crucial point is a consequence of

(4.10)–(4.12), which imply that Im $F_+(s) = 0$ for $s > s_0$. In order to turn (B.2) into a $N/D$ representation, we need the set $\{p_i\}$ of poles of $D$, as well as the set $\{z_j\}$ of its zeros on $[4, s_0]$, and define

$$\bar{D}(s) = \prod_i (s - p_i) \prod_j \frac{1}{s - z_j} D(s). \qquad \text{(B.4)}$$

To simplify the argument, we assume in the following that the $N$-functions $n$ and $n'$ have no zeros on $[4, s_0]$ – the discussion is more involved and the definitions (B.5) have to be modified if this is not the case. The $N/D$ representation of $f'$ is then obtained by eliminating $D$ in favor of $\bar{D}$ in the expression (B.2), and writing

$$n'(s) = Cn(s)\bar{D}(s); \quad d'(s) = Cd(s)\bar{F}(s), \qquad \text{(B.5)}$$

where $C$ is a constant and where

$$\begin{aligned}\bar{F}(s) = \bar{D}(s) + (s-4)H(s)f(s) \\ \times \prod_j \frac{1}{s - z_j} \prod_i (s - p_i).\end{aligned} \qquad \text{(B.6)}$$

The functions $n'$ and $d'$ defined in (B.5) have the correct analyticity properties. Denominator functions are normalized to one at infinity [10]. We have checked in our examples that $\bar{F}$ has a finite, nonzero limit at infinity. Therefore, $d'$ is properly normalized if $C = 1/\bar{F}(\infty)$. For $\delta(s_0) > 0$, $d$ can be written as

$$d(s) = \prod_{k=1}^{r} \left(\frac{s - s_0}{s - s_k}\right) \exp\left[-\frac{1}{\pi} \int_4^{s_0} \mathrm{d}x \frac{\delta(x)}{x - s}\right], \quad \text{(B.7)}$$

where the $s_k$ are the $r$ CDD poles of $f$, $s_k \in (4, s_0), r = [\delta(s_0)/\pi]$. We see that $d'$ is finite at $s_k$ if $s_k$ coincides with a zero $z_j$ of $D$, and if $\bar{F}(z_j) = 0$. Otherwise, every $s_k$ and $z_j$ is a CDD pole of $f'$. Within our assumptions on $n$ and $n'$, the fate of the CDD poles is dictated by the zeros of $D$: these produce new CDD poles in $f'$ or remove CDD poles present in $f$. The poles of $D$ are points where $f' = f$.

## C Proof of proposition 4

Using the framework of Sect. 4, we show that $f'$ coincides with $f$, if Re $f'_+$ is regular at $s_0$ [in the sense that it has a holomorphic extension into a circle of radius $\epsilon$ and center $s_0$]. See Fig. 9 for the analyticity domains used in the proof.

(i) The dispersion relation (2.1) written for $f'$ determines Re $f'_+$ on $\mathbb{R}$. Inversion of this relation gives

$$\mathrm{Im}\ f'_+(s) = -(s-4)\frac{1}{\pi} ⨍_{\mathbb{R}} \mathrm{d}x \frac{\phi(x)}{x - s}, \qquad \text{(C.1)}$$

where $\phi(s) = (\mathrm{Re}\ f'_+(s) - f'_+(4))/(s-4)$. The holomorphy of Re $f'_+$ in $C$ implies the holomorphy of Im $f'_+$ in the same circle.

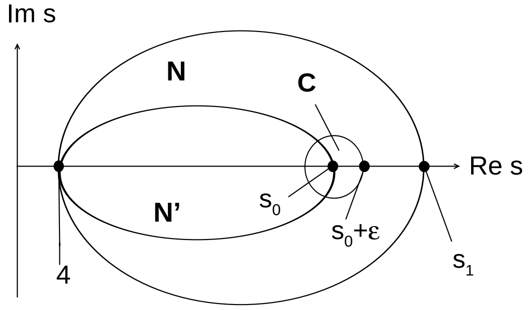

**Fig. 9.** Analyticity domains referred to in the proof of proposition 4

(ii) According to proposition 3, Im $f'_+$ has a holomorphic extension into a neighborhood $N'$ of $(4, s_0)$, with $s_0$ on its boundary. Combining this with the previous result we see that Im $f'_+$ is holomorphic in $\bar{N} = N' \cup C$, a domain extending up to $s_0 + \epsilon$.

(iii) Proposition 3 tells us that the high-energy absorptive part $A$, originally defined on $[s_0, \infty)$, has an analytic continuation into a neighborhood N of $(4, s_1)$, which coincides with Im $f_+$ on $[4, s_0]$. As $f'$ is a solution of the Roy equation with input $(a, A)$, we have

$$\mathrm{Im}\ f'_+(s) = A(s) \qquad \text{(C.2)}$$

for $s_0 \le s \le s_0 + \epsilon$. In view of this equality and of the regularity of Im $f'_+$ in $\bar{N}$, Im $f'_+$ has to be equal to $A$ on $[4, s_0]$. As $A$ is equal to Im $f_+$ on that interval, we conclude that $f' = f$.

(iv) Similarly, if Im $f'_+$ is assumed to be regular at $s_0$, one concludes that Re $f_+$ is regular at $s_0$ and $f' = f$. Therefore, $f'$ has to be singular at $s_0$ if $f' \neq f$. This is the content of proposition 4.

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